3D Reflection from a Mirror

Expressions for the direction of the reflected ray and points on the reflected beam path.

Serving the Intellectually Curious

THORLABS

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Tracing the Reflected Beam Path

- Each mirror in a setup has its own, independently adjustable, angular orientation.
- The beam path depends on each mirror's orientation.
- Defining a fixed (global) coordinate system for the setup is useful for tracing the beam path through the setup.
- However, it is simplest to calculate the direction of the reflected beam when working in the local coordinate system of the reflective surface.

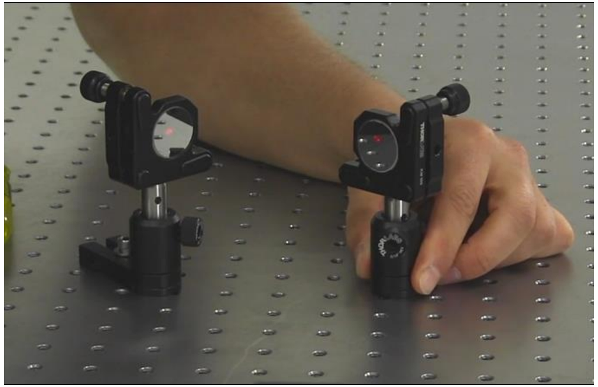


Figure 1. The beam path reflected by these two mirrors depends on their orientations with respect to one another.

- Therefore, both local and global coordinate systems are often used, and it is necessary to convert between them.

Surface Reflection in Terms of Local Coordinates

The incident ray reflects across the surface normal.

- The angles of incidence and reflection with the surface normal are the same.

$$\alpha_i = \alpha_r$$

- The optical angle (θ) between the incident and reflected rays is twice the angle between the incident ray and surface normal.

$$\theta = 2\alpha_i$$

- Changing the angle of incidence by an angle δ changes the optical angle (θ) between the incident and reflected rays by an angle 2δ.

$$\theta = 2(\alpha_i + \delta) = 2\alpha_i + 2\delta$$

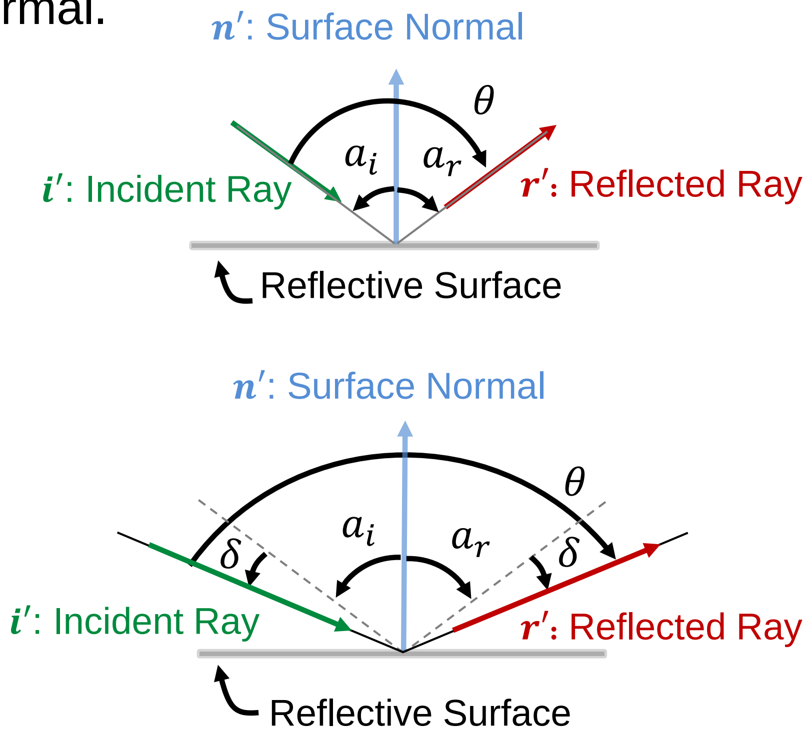


Figure 2. Incident and reflected ray angles with the surface normal are equal.

Surface Reflection in Terms of Local Coordinates

Calculating the reflected ray's coordinates.

- In the local coordinate system, the surface is in the plane of the u'- and v'-axes, while the w'-axis is normal to the surface and u'-v' plane.

- The incident and reflected rays have unit vectors, i' and r', respectively, in which:

  $i'_{\parallel} = r'_{\parallel}$ Components Parallel to *u*'-*v*' Plane

  $i'_{\perp} = -r'_{\perp}$ Components Perpendicular to *u*'-*v*' Plane

- The reflected ray ($\boldsymbol{r}'$) in local coordinates, is calculated by reflecting the incident ray ($\boldsymbol{i}'$) across the surface normal ($\boldsymbol{n}'$):

$$\boldsymbol{r}' = \boldsymbol{i}' - 2(\boldsymbol{i}' \cdot \boldsymbol{n}')\boldsymbol{n}'$$

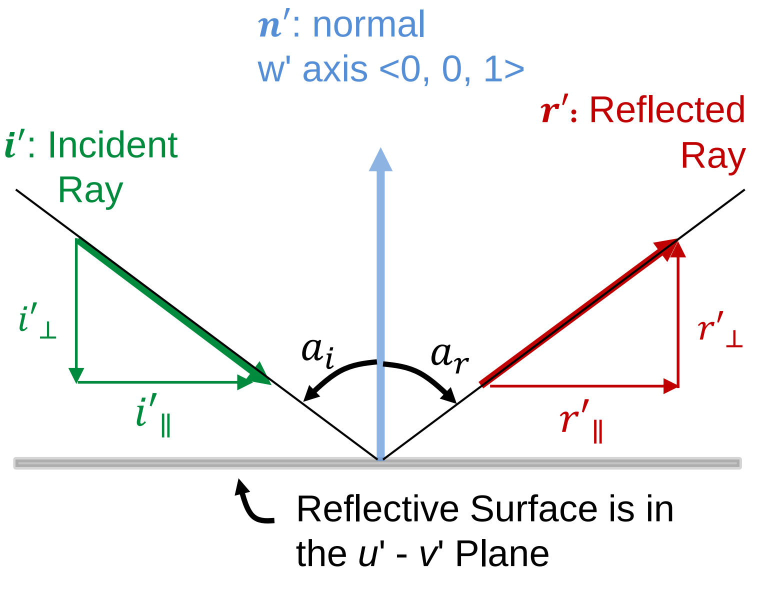


Figure 3. The angle between the incident ray and surface normal equals the angle between the reflected ray and surface normal.

The Different Local and Global Perspectives

Identifying differences in and relating local and global system perspectives.

- Local perspective: the surface never moves. Instead, the angle of the incident ray changes.
- Global perspective: the surface rotates relative to the incident ray.
- Example: choose a global coordinate system and define the orientation of the unrotated reflective surface.
  - Position of surface center: (0, 0, 0).
  - The unrotated surface is in the *x*-*y* plane.
  - The *z*-axis is normal to the unrotated surface.

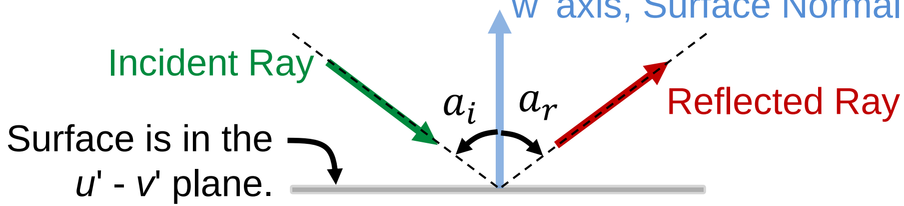


Figure 4. Local Perspective of Surface

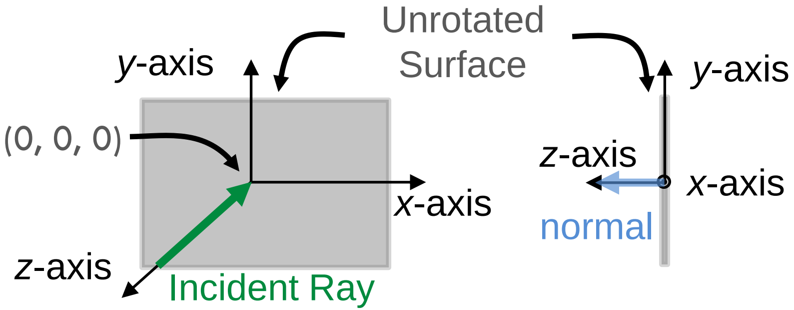


Figure 5. First step to a global view is defining the orientation of the unrotated surface.

Define Some Global Coordinate System Conventions

Define global coordinate conventions for rotating the surface.

- Global coordinates include information about the rotation angles of the surface relative to the global axes.
- Pitch and Yaw Rotation of the Reflective Surface
  - Positive pitch (θ) around *x*-axis is counterclockwise (CCW), when looking towards the origin, down the *x*-axis.
  - Positive yaw (ψ) around y-axis is CCW, when looking towards the origin, down the *y*-axis.
  - The center of the reflective surface always coincides with the origin, regardless of its orientation.

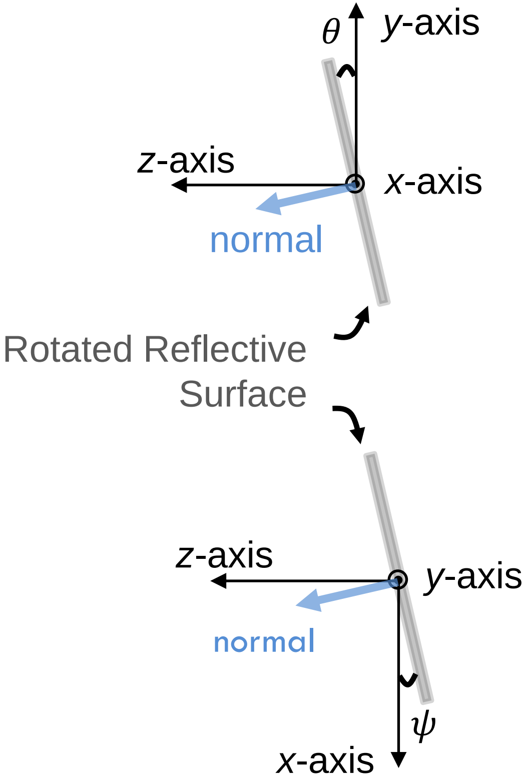


Figure 6. Rotation angles with respect to the *x*- and *y*-axis (θ and ψ, respectively) are measured counterclockwise.

Reflection Matrices: Coordinates and Unit Vectors

Points on the surface have local and global coordinates and unit vectors.

- Both points and vectors can be converted between local and global coordinate systems.
- Unit vectors directed from the origin, towards a point:
  - Local unit vector: $\boldsymbol{r'} = \langle u', v', w' \rangle$
  - Global unit vector: $\boldsymbol{r} = \langle x, y, z \rangle$
  - For the unrotated surface, $\boldsymbol{r'} = \boldsymbol{r}$.
- Points on the surface:
  - Local coordinates: $P' = (u', v', w')$
  - Global coordinates: $P = (x, y, z)$
  - For the unrotated surface, $P' = P$.

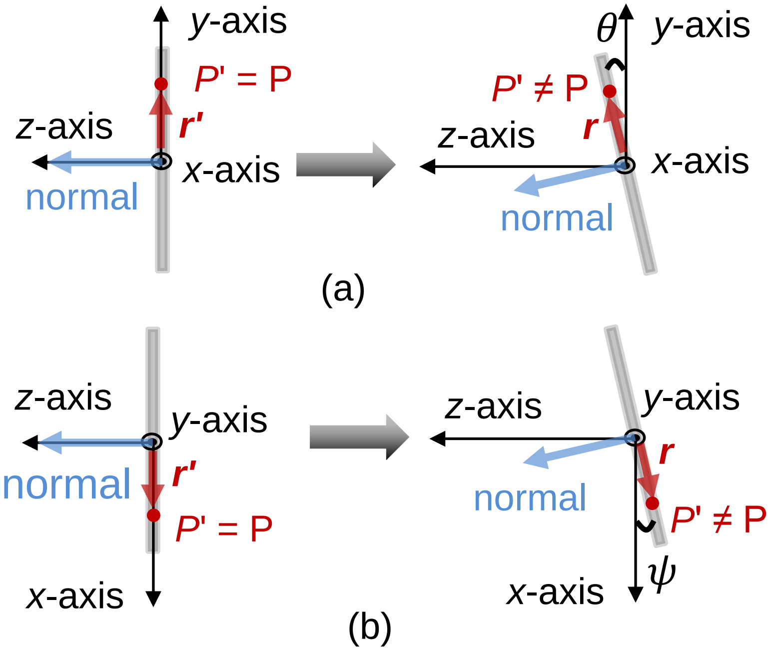


Figure 7. Rotation around the (a) *x*-axis and (b) *y*-axis. Unrotated surface is on the left, rotated surface is on the right.

Reflection Matrices: Unrotated to Rotated Orientations

Matrix algebra can be used to rotate vectors and points around an axis.

- Converting a local point or unit vector to global coordinates requires including information about the rotation angles. This can be done using matrices.
- When rotation is CCW around the *x*-axis:

$$\boldsymbol{P} = \boldsymbol{R}_x(\theta)\boldsymbol{P}'$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

- When rotation is CCW around the *y*-axis:

$$\boldsymbol{P} = \boldsymbol{R}_y(\psi)\boldsymbol{P}'$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

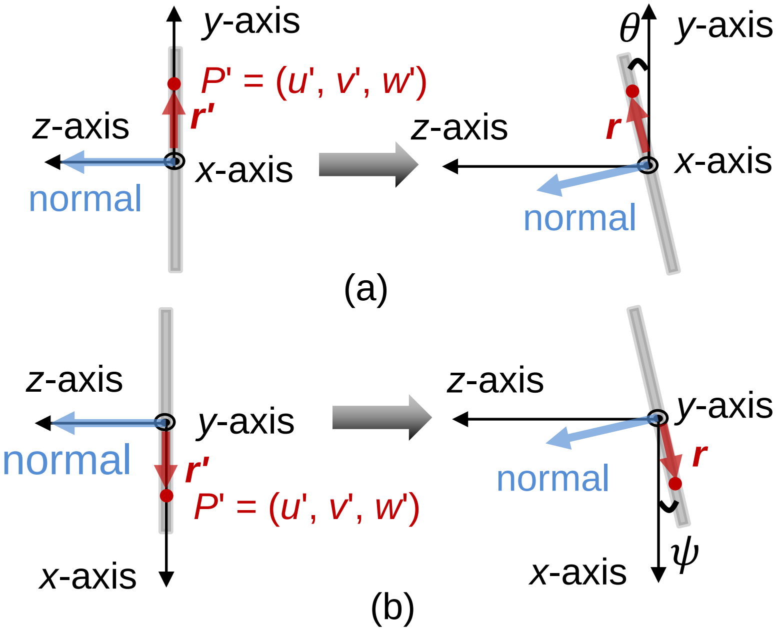
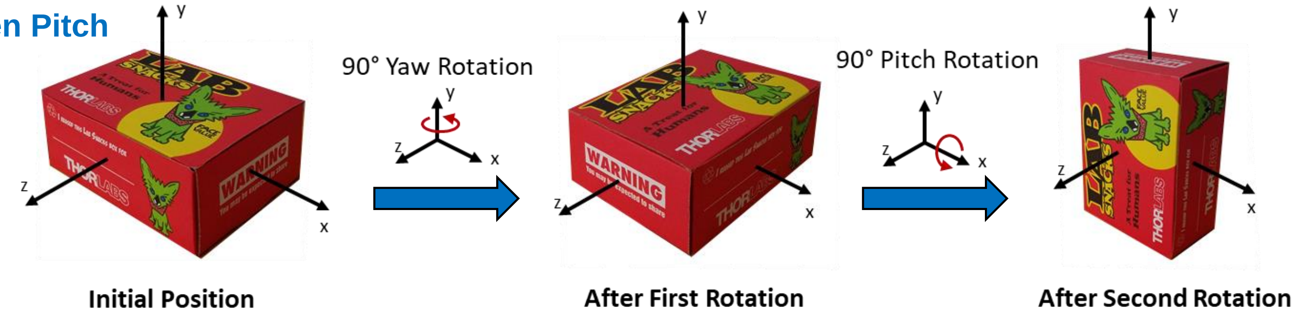


Figure 8. Counterclockwise (CCW) rotation around the (a) *x*-axis and (b) *y*-axis. Unrotated surface is on the left, rotated surface is on the right.

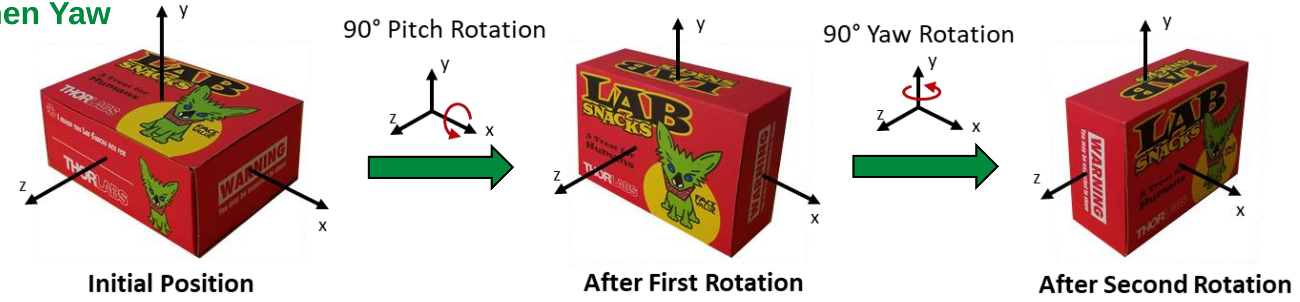
Object Orientation Depends on Order of Rotations

Note that when rotating an object around a coordinate system's axes, the object's final orientation depends on the order in which the rotations were performed.

Case 1:
Yaw then Pitch



Case 2:
Pitch then Yaw



Object Orientation Depends on Order of Rotations

In other words, rotation matrices are not commutative.

- The final orientation of the reflective surface depends on the order in which the mirror's pitch and yaw axes were adjusted.
- Therefore, the order in which the pitch and yaw axes are adjusted determines:
  - The orientation (direction) of the reflected beam.
  - The beam path.
- To ensure agreement between experimental and modeled results, the order, direction, and magnitude of the rotations in the experimental and modeled cases must be in perfect agreement.

Reflection Matrices: Sequence of Rotations

A single, custom matrix converts between local and global coordinate systems.

- A sequence of rotations is typically used to orient a mirror. A total matrix ($\boldsymbol{R}_{Total}$) converts between coordinate systems and accounts for all applied rotations.
- Compute the total matrix by multiplying the individual rotation matrices together. Multiply them in the order in which the sequence of rotations was performed.
- For example, if the first rotation was around the *x*-axis ($\boldsymbol{R}_x(\theta)$), and the second was around the *y*-axis ($\boldsymbol{R}_y(\psi)$), the rotation matrix ($\boldsymbol{R}_{yx}(\theta,\psi)$) is the product:

$$\boldsymbol{P} = \boldsymbol{R}_{\boldsymbol{Total}}\boldsymbol{P}' = \boldsymbol{R}_y(\psi)\boldsymbol{R}_x(\theta)\boldsymbol{P}' = \boldsymbol{R}_{yx}(\theta,\psi)\boldsymbol{P}'$$

$$\boldsymbol{R}_{yx}(\theta,\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\theta\sin\psi & \cos\theta\sin\psi \\ 0 & \cos\theta & -\sin\theta \\ -\sin\psi & \sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix}$$

For information about matrix rotations and transformations, refer to a linear algebra reference, such as: D. Cherney, T. Denton, R. Thomas, and A. Waldron(Davis, CA 2013). *Linear Algebra*. https://www.math.ucdavis.edu/~linear/linear-guest.pdf

Calculate the Reflected Ray's Global Coordinates

Procedure for calculating the reflected ray's direction relative to the incident ray's.

- Transform the unit vector of the incident ray from global ($\boldsymbol{i}$) to local ($\boldsymbol{i}'$) coordinates using the inverse rotation matrix ($\boldsymbol{R}_{Total}^{-1}$):

$$\boldsymbol{i}' = \boldsymbol{R}_{Total}^{-1}(\theta, \psi)\boldsymbol{i}$$

- Reflect the ray across the local normal ($\boldsymbol{n}'$) to obtain the reflected ray ($\boldsymbol{r}'$) in local coordinates:

$$\boldsymbol{r}' = \boldsymbol{i}' - 2(\boldsymbol{i}' \cdot \boldsymbol{n}')\boldsymbol{n}'$$

- Convert the reflected ray back into global coordinates ($\boldsymbol{r}$) using the rotation matrix ($\boldsymbol{R}_{total}$):

$$\boldsymbol{r} = \boldsymbol{R}_{Total}(\theta, \psi)\boldsymbol{r}'$$

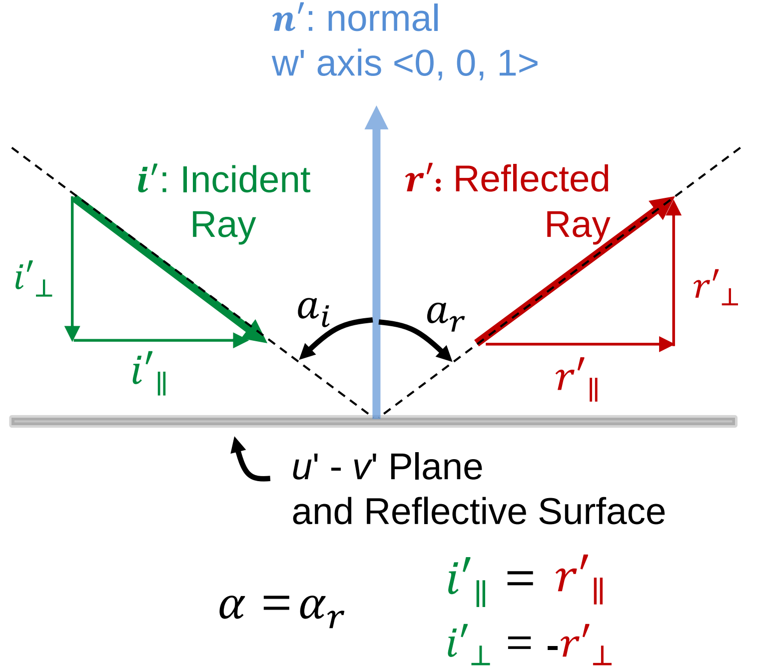


Figure 9. Reflection is performed using local coordinates, which involves reflecting the incident ray across the normal to the *u*-*v* plane.

Example 1: Mount's Adjusters Tune the Reflected Beam's Direction

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Example 1 Overview: Adjusters Tune Orientation

Find the reflected ray's direction, relative to the incident ray, after angle tuning.

- The mount's back plate does not move.
- The mirror is installed in the mount's front plate, which can rotate around the pivot point.
- The mirror's orientation is tuned using only the mount's adjusters, which rotate the mount's front plate about the pivot point.
- The mirror rotates relative to the fixed global *x*-, *y*-, and *z*-axes, whose origin is chosen to be the front plate's pivot point.
- The incident ray's direction is fixed and parallel to the *z*-axis.

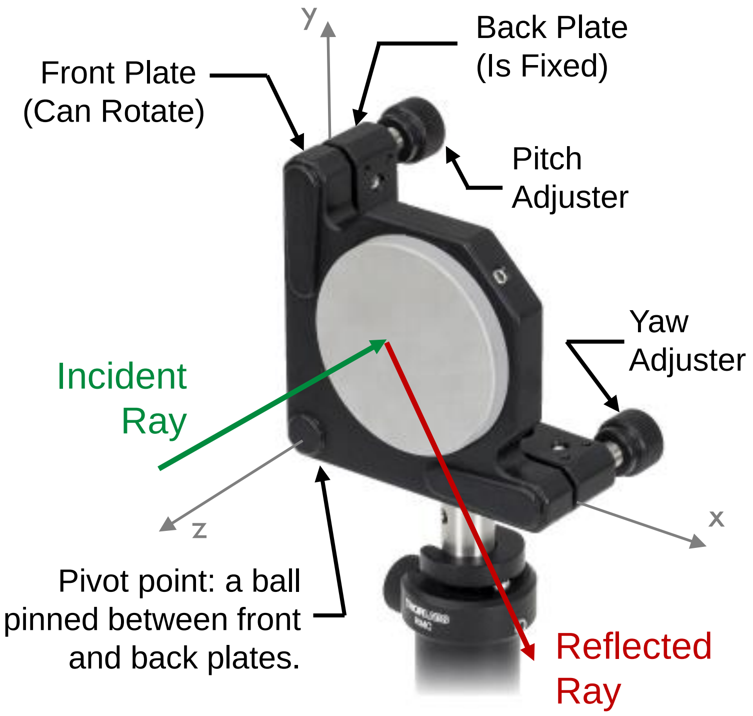


Figure 10. Mirror mounted in a KM200 kinematic mirror mount.

Example 1 Overview: Adjusters Tune Orientation

Two rotations, one pitch and one yaw, were performed in succession.

- The mount's back plate cannot move, since it is secured to a post, which is clamped in a post holder.
- The pitch and yaw adjusters are installed in the back plate.
- The adjusters' tips push against the backside of the front plate.
  - Tuning the pitch adjuster rotates the front plate around the *x*-axis.
  - Tuning the yaw adjuster rotates the front plate around the *y*-axis.

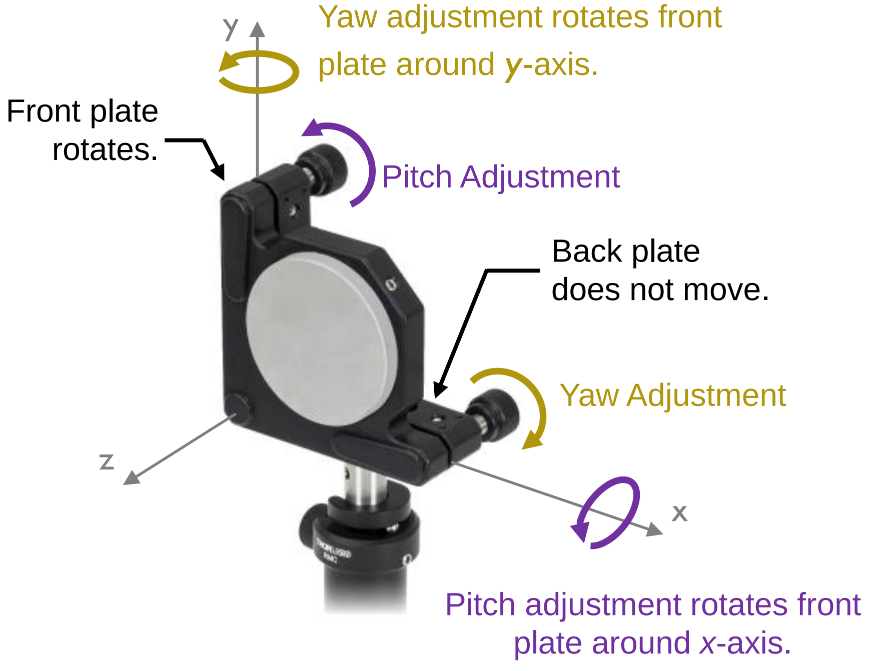


Figure 11. Adjusters can be used to tune the mirror's pitch and yaw.

Calculate the Reflected Ray's Global Coordinates

Procedure for calculating the reflected ray's direction relative to the incident ray's.

- Transform the unit vector of the incident ray from global ($\boldsymbol{i}$) to local ($\boldsymbol{i}'$) coordinates using the inverse rotation matrix ($\boldsymbol{R}_{Total}^{-1}$):

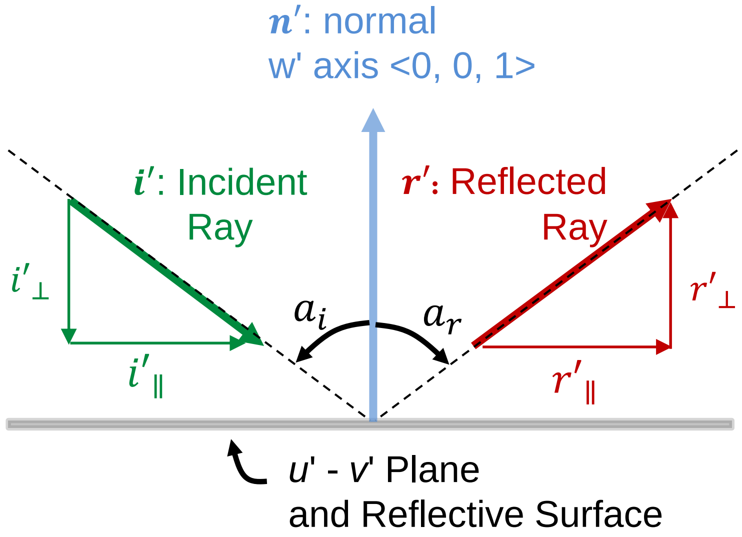
$$\boldsymbol{i}' = \boldsymbol{R}_{Total}^{-1}(\theta,\psi)\boldsymbol{i}$$

- Reflect the ray across the local normal ($\boldsymbol{n}'$) to obtain the reflected ray ($\boldsymbol{r}'$) in local coordinates:

$$\boldsymbol{r}' = \boldsymbol{i}' - 2(\boldsymbol{i}' \cdot \boldsymbol{n}')\boldsymbol{n}'$$

- Convert the reflected ray back into global coordinates ($\boldsymbol{r}$) using the rotation matrix ($\boldsymbol{R}_{total}$):

$$\boldsymbol{r} = \boldsymbol{R}_{Total}(\theta,\psi)\boldsymbol{r}'$$



$$\alpha = \alpha_r \qquad i'_{\parallel} = r'_{\parallel} \qquad i'_{\perp} = -r'_{\perp}$$

Figure 12. Reflection is performed using local coordinates, which involves reflecting the incident ray across the normal to the *u*-*v* plane.

Mount Adjusters Tuned: Local to Global Transformations

- The total rotation matrix ($\boldsymbol{R}_{Total}(\theta,\psi)$), which converts **local** (unrotated) to **global** (rotated) coordinates, differs depending on whether pitch or yaw is adjusted first.

$$\boldsymbol{P}_{Global} = \boldsymbol{R}_{Total}(\theta,\psi)\boldsymbol{P'}_{Local}$$

If yaw, then pitch, was adjusted, the local to global transformation:

$$\boldsymbol{R}_{xy}(\theta,\psi) = \boldsymbol{R}_x(\theta)\boldsymbol{R}_y(\psi)$$

Pitch / Yaw

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}$$

$$\boldsymbol{R}_{xy}(\theta,\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ \sin\theta\sin\psi & \cos\theta & -\sin\theta\cos\psi \\ -\cos\theta\sin\psi & \sin\theta & \cos\theta\cos\psi \end{bmatrix}$$

If pitch, then yaw, was adjusted, the local to global transformation:

$$\boldsymbol{R}_{yx}(\psi,\theta) = \boldsymbol{R}_y(\psi)\boldsymbol{R}_x(\theta)$$

Yaw / Pitch

$$= \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\boldsymbol{R}_{yx}(\psi,\theta) = \begin{bmatrix} \cos\psi & \sin\theta\sin\psi & \cos\theta\sin\psi \\ 0 & \cos\theta & -\sin\theta \\ -\sin\psi & \sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix}$$

Mount Adjusters Tuned: Global to Local Transformations

The inverse total rotation matrix ($\boldsymbol{R}_{Total}^{-1}(\theta,\psi)$) converts global to local coordinates.

- **Global** (rotated) to **local** (unrotated) coordinates: $\boldsymbol{P'}_{Local} = \boldsymbol{R}^{-1}(\theta,\psi)\boldsymbol{P}_{Global}$
  - $\boldsymbol{R}^{-1}(\theta,\psi)$ is the inverse of $\boldsymbol{R}(\theta,\psi)$.
  - In the case of rotation matrices, the inverse equals the transpose.

If yaw, then pitch, was adjusted, the global to local transformation:

$$\boldsymbol{R}_{xy}^{-1}(\theta,\psi) = \begin{bmatrix} \cos\psi & \sin\theta\sin\psi & -\cos\theta\sin\psi \\ 0 & \cos\theta & \sin\theta \\ \sin\psi & -\sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix}$$

If pitch, then yaw, was adjusted, the global to local transformation:

$$\boldsymbol{R}_{yx}^{-1}(\psi,\theta) = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ \sin\theta\sin\psi & \cos\theta & \sin\theta\cos\psi \\ \cos\theta\sin\psi & -\sin\theta & \cos\theta\cos\psi \end{bmatrix}$$

Mount Adjusters Tuned: Incident Ray in Local Coords

Start by transforming the unit vector of the incident ray into local coordinates.

- Since incident ray is parallel to the z-axis, the unit vector of the incident ray in global coordinates is: $\boldsymbol{i} = \langle 0, 0, -1 \rangle = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

If yaw, then pitch, was adjusted, the incident ray in local coordinates:

$$\boldsymbol{i}'_{Local} = \boldsymbol{R}_{xy}^{-1}(\theta, \psi)\boldsymbol{i}_{Global}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\theta\sin\psi & -\cos\theta\sin\psi \\ 0 & \cos\theta & \sin\theta \\ \sin\psi & -\sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\cos\theta\sin\psi \\ -\sin\theta \\ -\cos\theta\cos\psi \end{bmatrix}$$

If pitch, then yaw, was adjusted, the incident ray in local coordinates:

$$\boldsymbol{i}'_{Local} = \boldsymbol{R}_{yx}^{-1}(\psi, \theta)\boldsymbol{i}_{Global}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ \sin\theta\sin\psi & \cos\theta & \sin\theta\cos\psi \\ \cos\theta\sin\psi & -\sin\theta & \cos\theta\cos\psi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\sin\psi \\ -\sin\theta\cos\psi \\ -\cos\theta\cos\psi \end{bmatrix}$$

Mount Adjusters Tuned: Reflected Ray in Local Coords

Calculate the unit vector of the reflected ray in local coordinates.

- Use the incident ray, in local coordinates, to calculate the reflected ray in local coordinates:

$$\boldsymbol{r}'_{Local} = \boldsymbol{i}'_{Local} - 2(\boldsymbol{i}'_{Local} \cdot \boldsymbol{n}'_{Local})\boldsymbol{n}'_{Local}$$

If yaw, then pitch, was adjusted, the reflected ray in local coordinates:

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\cos\theta\sin\psi \\ -\sin\theta \\ -\cos\theta\cos\psi \end{bmatrix} - 2\left\{\begin{bmatrix} +\cos\theta\sin\psi \\ -\sin\theta \\ -\cos\theta\cos\psi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\cos\theta\sin\psi \\ -\sin\theta \\ +\cos\theta\cos\psi \end{bmatrix}$$

If pitch, then yaw, was adjusted, the reflected ray in local coordinates:

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\sin\psi \\ -\sin\theta\cos\psi \\ -\cos\theta\cos\psi \end{bmatrix} - 2\left\{\begin{bmatrix} +\sin\psi \\ -\sin\theta\cos\psi \\ -\cos\theta\cos\psi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\sin\psi \\ -\sin\theta\cos\psi \\ +\cos\theta\cos\psi \end{bmatrix}$$

Mount Adjusters Tuned: Reflected Ray in Global Coords

Calculate the unit vector of the reflected ray in global coordinates.

- Transform the reflected ray from local coordinates into global coordinates:

If yaw, then pitch, was adjusted, the reflected ray in global coordinates:

$$\boldsymbol{r}_{Global} = \boldsymbol{R}_{xy}(\theta,\psi)\boldsymbol{r'}_{Local}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ \sin\theta\sin\psi & \cos\theta & -\sin\theta\cos\psi \\ -\cos\theta\sin\psi & \sin\theta & \cos\theta\cos\psi \end{bmatrix} \begin{bmatrix} +\cos\theta\sin\psi \\ -\sin\theta \\ +\cos\theta\cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos\theta\cos\psi\ \sin\psi \\ -2\cos\theta\sin\theta\cos^2\psi \\ -\cos^2\theta\sin^2\psi - \sin^2\theta + \cos^2\theta\cos^2\psi \end{bmatrix}$$

If pitch, then yaw, was adjusted, the reflected ray in global coordinates:

$$\boldsymbol{r}_{Global} = \boldsymbol{R}_{xy}(\theta,\psi)\boldsymbol{r'}_{Local}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\theta\sin\psi & \cos\theta\sin\psi \\ 0 & \cos\theta & -\sin\theta \\ -\sin\psi & \sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix} \begin{bmatrix} +\sin\psi \\ -\sin\theta\cos\psi \\ +\cos\theta\cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\ \sin\psi\,(1 - \sin^2\theta + \cos^2\theta) \\ -2\cos\theta\sin\theta\cos\psi \\ -\sin^2\psi - \sin^2\theta\cos^2\psi + \cos^2\theta\cos^2\psi \end{bmatrix}$$

Mount Adjusters Tuned: Reflected Ray in Global Coords

Simplify the z-component for the reflected ray in global coordinates.

- If yaw, then pitch, was adjusted, the reflected vector in global coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos\theta\cos\psi\ \sin\psi \\ -2\cos\theta\sin\theta\cos^2\psi \\ -\cos^2\theta\sin^2\psi - \sin^2\theta + \cos^2\theta\cos^2\psi \end{bmatrix}$$

$$= -\cos^2\theta\sin^2\psi - \sin^2\theta + \cos^2\theta\cos^2\psi + (\cos^2\theta\cos^2\psi - \cos^2\theta\cos^2\psi)$$

$$= -(\cos^2\theta\,\{\cos^2\psi + \sin^2\psi\} + \sin^2\theta) + 2\cos^2\theta\cos^2\psi$$

$$= -(\cos^2\theta\,\{1\} + \sin^2\theta) + 2\cos^2\theta\cos^2\psi$$

$$= -1 + 2\cos^2\theta\cos^2\psi$$

Mount Adjusters Tuned: Reflected Ray in Global Coords

Simplify the x- and z-components for the reflected ray in global coordinates.

- If pitch, then yaw, was adjusted, the reflected vector in global coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\psi\ \sin\psi\,(1 - \sin^2\theta + \cos^2\theta) \\ -2\cos\theta\sin\theta\cos^2\psi \\ -\sin^2\psi - \sin^2\theta\cos^2\psi + \cos^2\theta\cos^2\psi \end{bmatrix}$$

$$= \cos\psi\ \sin\psi\,(\cos^2\theta + \cos^2\theta)$$
$$= 2\cos^2\theta\cos\psi\ \sin\psi$$

$$= -1 + \cos^2\psi - \sin^2\theta\cos^2\psi + \cos^2\theta\cos^2\psi$$
$$= -1 + \cos^2\psi\,(\cos^2\theta + \sin^2\theta) - \sin^2\theta\cos^2\psi + \cos^2\theta\cos^2\psi$$
$$= -1 + 2\cos^2\theta\cos^2\psi$$

Mount Adjusters Tuned: Reflected Ray in Global Coords

Expression for the unit vector of the reflected ray in global coordinates.

- The z-component is the same in both cases, but the x- and y-components differ.

If first the mirror's yaw (ψ) and then its pitch (θ) was adjusted, the reflected ray in global coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos\theta\cos\psi\sin\psi \\ -2\cos\theta\sin\theta\cos^2\psi \\ -1 + 2\cos^2\theta\cos^2\psi \end{bmatrix}$$

If first the mirror's pitch (θ) and then its yaw (ψ) was adjusted, the reflected ray in global coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos^2\theta\cos\psi\ \sin\psi \\ -2\cos\theta\sin\theta\cos\psi \\ -1 + 2\cos^2\theta\cos^2\psi \end{bmatrix}$$

Mount Adjusters Tuned: Using the Reflected Ray

Use the reflected ray's unit vector to calculate arbitrary points on the ray path.

- The reflected ray's unit vector $\langle x, y, z \rangle$ is useful because,
  - It points in the direction of the ray's path.
  - It has a length of 1, so the point (x, y, z) lies at the ray's tip: $\sqrt{x^2 + y^2 + z^2} = 1$.
  - It can be used to calculate the coordinates of any point on the beam path.
- To calculate an arbitrary point (x_2, y_2, z_2) on the reflected ray's path,
  - Calculate a new vector by multiplying the unit vector by a constant *A*:

$$\mathrm{A}\langle x, y, z \rangle = \langle Ax, Ay, Az \rangle$$

  - The length of the new vector is *A*: $\sqrt{|Ax|^2 + |Ay|^2 + |Az|^2} = A$
  - Choose the length so that: $(x_2, y_2, z_2) = (Ax, Ay, Az)$
  - The point (x_2, y_2, z_2) lies at the new vector's tip.

Mount Adjusters Tuned: Points on the Reflected Ray

Calculating an arbitrary point on the beam path when another point is known.

- The known point is: (x_1, y_1, z_1).
- The coordinates of the known point can be added to a vector equal to the reflected unit vector whose length has been scaled by the correct value of A.

If yaw (ψ), then pitch (θ), was adjusted, the unit vector of the reflected ray:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos\theta\cos\psi\sin\psi \\ -2\cos\theta\sin\theta\cos^2\psi \\ -1 + 2\cos^2\theta\cos^2\psi \end{bmatrix}$$

Points on the reflected ray:

$$x_2 = x_1 + A(2\cos\theta\cos\psi\sin\psi)$$
$$y_2 = y_1 + A(-2\cos\theta\sin\theta\cos^2\psi)$$
$$z_2 = z_1 + A(-1 + 2\cos^2\theta\cos^2\psi)$$

If pitch (θ), then yaw (ψ), was adjusted, the unit vector of the reflected ray:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos^2\theta\cos\psi\ \sin\psi \\ -2\cos\theta\sin\theta\cos\psi \\ -1 + 2\cos^2\theta\cos^2\psi \end{bmatrix}$$

Points on the reflected ray:

$$x_2 = x_1 + A(2\cos^2\theta\cos\psi\sin\psi)$$
$$y_2 = y_1 + A(-2\cos\theta\sin\theta\cos\psi)$$
$$z_2 = z_1 + A(-1 + 2\cos^2\theta\cos^2\psi)$$

Mount Adjusters Tuned: Points on the Reflected Ray

Assume the known point is at the origin: (x_1, y_1, z_1) = (0, 0, 0).

- Calculate the required spacing between paired steering mirrors.
  - Option 1: Vary the scaling factor (*A*) to change the distance from origin to beam point.
  - Option 2: Use a known point coordinate (e.g. a new beam height y_2), the pitch angle, and the yaw angle to calculate A. Then, calculate the other two point coordinates.

If yaw (ψ), then pitch (θ), was adjusted, the separation between the origin and the point (x_2, y_2, z_2) on the reflected ray:

$$x_2 = A(2\cos\theta\cos\psi\sin\psi)$$
$$y_2 = A(-2\cos\theta\sin\theta\cos^2\psi)$$
$$z_2 = A(-1 + 2\cos^2\theta\cos^2\psi)$$

If pitch (θ), then yaw (ψ), was adjusted, the separation between the origin and the point (x_2, y_2, z_2) on the reflected ray:

$$x_2 = A(2\cos^2\theta\cos\psi\sin\psi)$$
$$y_2 = A(-2\cos\theta\sin\theta\cos\psi)$$
$$z_2 = A(-1 + 2\cos^2\theta\cos^2\psi)$$

Example 2: Mount Rotation for Yaw, Adjuster Tuning for Pitch

Example 2 Overview: Choose Global Coordinates

Define a fixed, global coordinate system (*x*-, *y*-, and *z*-axes).

- The chosen fixed and global coordinate system:
  - The *x*-*y* plane is in the plane of the unrotated mirror.
  - The *z*-axis is aligned with the incident ray, whose orientation is fixed.
  - The global coordinate system's origin is placed at the center of the unrotated mirror's surface.
- The directions of the incident and reflected rays are defined relative to these fixed, global coordinate axes.
- Rotating the mirror moves it relative to the global coordinate system.

Figure 13. The *x*-, *y*-, and *z*-axes of the global coordinate system.

Example 2 Overview: Rotate Entire Mount to Change Yaw

Effect of rotating the mount around the post axis.

- Rotation around the post axis is an alternative to using the yaw adjuster.
  - Both front and back plates rotate together as one rigid unit.
  - The yaw adjuster would have rotated the front plate rotate to the back plate.

- The effect of rotating the mount:
  - The reflective surface of the rotated mirror is at an angle to the *x*-*y* plane.
  - The mirror's surface is no longer normal to the incident ray and *z*-axis.
  - From the mount's point of view, the direction of the incident ray has changed.

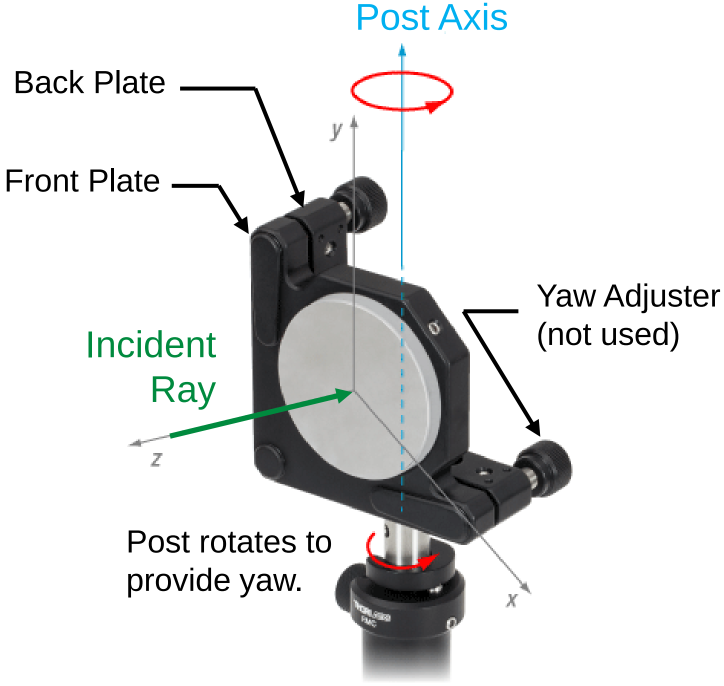


Figure 14. Entire mount is rotated around the post axis to change yaw.

Example 1 Overview: Use Mount's Adjuster to Tune Pitch

Procedure for and effect of changing the mirror's pitch.

- Pitch is adjusted by tuning the mount's pitch adjuster.
  - Mount's pitch adjuster is anchored in the back plate, which does not move.
  - Adjuster's tip presses against the front plate's backside, forcing it to rotate.
- Tuning pitch rotates the front plate around the *x*'-, *y*'-, *z*'-axes.
  - These axes are anchored to the mount's back plate, and the origin is the pivot point.
  - Front plate rotates around the axes' origin, relative to the back plate.

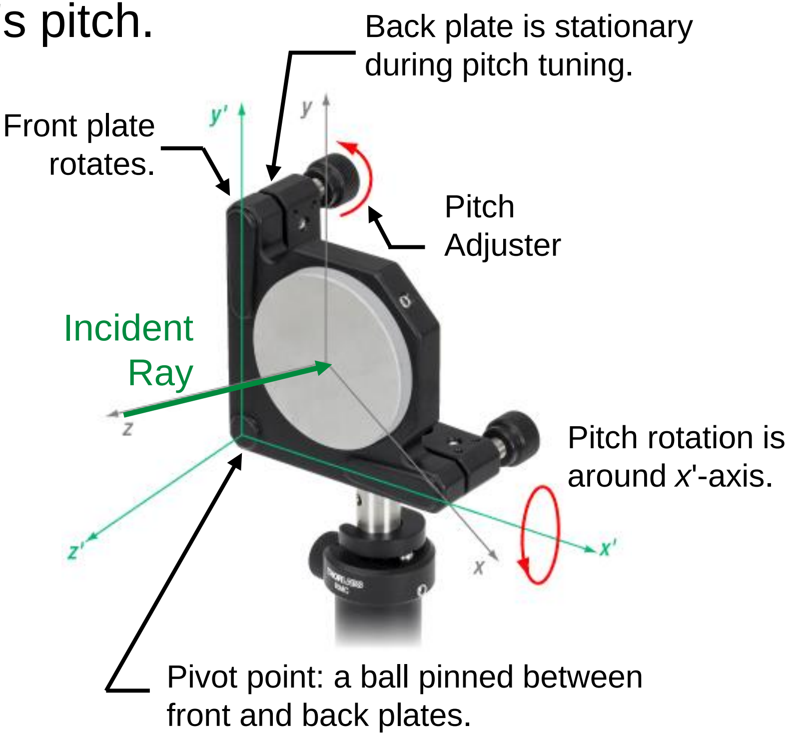


Figure 15. Mirror mounted in a KM200 kinematic mirror mount.

Problem Statement: Combining Rotation Types

Combining mount rotation around post axis with adjuster tuning of front plate.

- Effect of rotating the mount around the post axis to provide yaw:
  - Front plate's position relative to back plate remains the same.
  - Incident ray's angle of incidence with the mount changes.
- Effect of using mount adjusters.
  - Front plate moves relative to the back plate.
  - Incident ray's angle of incidence with respect to the mount does not change.
- Reflected ray's final direction is the same, whether post axis rotation precedes or follows adjuster tuning.
- Include the effect of the post axis rotation by converting the incident ray's unit vector into the mount's (x', y', z') coordinates.

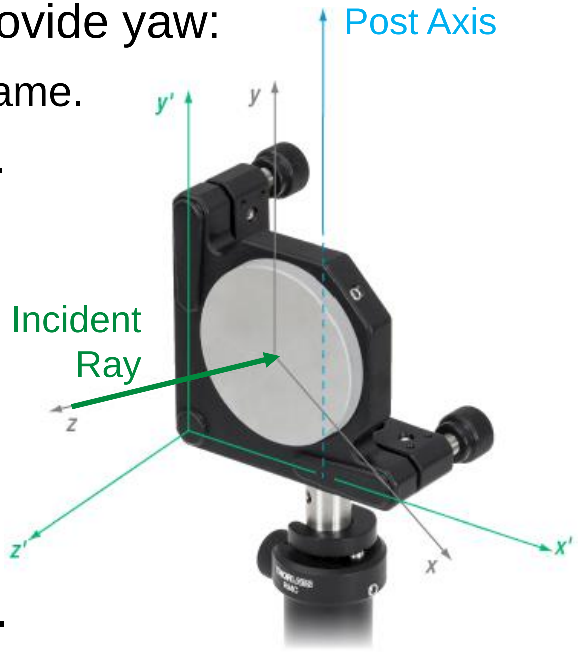


Figure 16. Rotation Axes

Calculate the Reflected Ray's Global Coordinates

Convert the incident ray to mount coordinates to account for post axis rotation.

- Convert incident ray's unit vector from global ($\boldsymbol{i}$) to mount ($\boldsymbol{i}'$) coordinates using the ($\boldsymbol{R}_y^{-1}$) matrix: $\boldsymbol{i}' = \boldsymbol{R}_y^{-1}(\phi)\boldsymbol{i}$
- Convert incident ray from mount ($\boldsymbol{i}'$) to mirror ($\boldsymbol{i}''$) coordinates using inverse rotation matrix ($\boldsymbol{R}_{Total}^{-1}$): $\boldsymbol{i}'' = \boldsymbol{R}_{Total}^{-1}(\theta,\psi)\boldsymbol{i}'$
- Reflect the ray across the local normal ($\boldsymbol{n}''$) to obtain the reflected ray ($\boldsymbol{r}''$) in local coordinates: $\boldsymbol{r}'' = \boldsymbol{i}'' - 2(\boldsymbol{i}'' \cdot \boldsymbol{n}'')\boldsymbol{n}''$
- Convert reflected ray from mirror ($\boldsymbol{r}''$) to mount ($\boldsymbol{r}'$) coordinates using the rotation matrix ($\boldsymbol{R}_{Total}$): $\boldsymbol{r}' = \boldsymbol{R}_{Total}(\theta,\psi)\boldsymbol{r}''$
- Convert the reflected ray from mount ($\boldsymbol{r}'$) to global ($\boldsymbol{r}$) coordinates using the rotation matrix ($\boldsymbol{R}_y$): $\boldsymbol{r} = \boldsymbol{R}_y(\phi)\boldsymbol{r}'$

Post Axis Rotation Combined with Pitch Adjuster Tuning

Example: Reflected ray orientated via post axis rotation and mount adjuster tuning.

- Global coordinates of incident ray's unit vector: $\boldsymbol{i} = \langle 0, 0, -1 \rangle = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

- Express the orientation of the incident ray's unit vector in the mount's coordinate system. Use the inverse yaw rotation matrix ($\boldsymbol{R}_y^{-1}(\phi)$), where ϕ is the CCW rotation angle of the mount around the post axis.

$$\boldsymbol{i}'_{Mount} = \boldsymbol{R}_y^{-1}(\phi)\boldsymbol{i}_{Global}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \sin\phi \\ 0 \\ -\cos\phi \end{bmatrix}$$

Post Axis Rotation Combined with Pitch Adjuster Tuning

Example: Reflected ray orientated via post axis rotation and mount adjuster tuning.

- Express the incident ray in the local coordinates of the reflective surface, as was done in Example 1.

- In this case, only the pitch adjuster was tuned ($\boldsymbol{R}_{Total}^{-1}(\theta,\psi) = \boldsymbol{R}^{-1}_{x}(\theta)$). Tuning the adjuster rotates the mirror CCW around the x'-axis by an angle θ.

$$\boldsymbol{i}''_{Local} = \boldsymbol{R}_{Total}^{-1}(\theta,\psi)\boldsymbol{i}'_{Mount} = \boldsymbol{R}^{-1}_{x}(\theta)\boldsymbol{i}'_{Mount}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\phi \\ 0 \\ -\cos\phi \end{bmatrix} = \begin{bmatrix} \sin\phi \\ -\sin\theta\cos\phi \\ -\cos\theta\cos\phi \end{bmatrix}$$

Post Axis Rotation Combined with Pitch Adjuster Tuning

Example: Reflected ray orientated via post axis rotation and mount adjuster tuning.

- Reflect the incident ray across the surface normal. The result is the unit vector of the reflected ray, expressed in the local coordinates of the mirror.

$$\boldsymbol{r}'' = \boldsymbol{i}'' - 2(\boldsymbol{i}'' \cdot \boldsymbol{n}'')\boldsymbol{n}''$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\sin\phi \\ -\sin\theta\cos\phi \\ -\cos\theta\cos\phi \end{bmatrix} - 2\left\{\left(\begin{bmatrix} +\sin\phi \\ -\sin\theta\cos\phi \\ -\cos\theta\cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right\} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} +\sin\phi \\ -\sin\theta\cos\phi \\ +\cos\theta\cos\phi \end{bmatrix}$$

Post Axis Rotation Combined with Pitch Adjuster Tuning

Example: Reflected ray orientated via post axis rotation and mount adjuster tuning.

- Express the reflected ray in the coordinates of the mount. Since only the pitch adjuster was tuned, $(\boldsymbol{R}_{Total}(\theta,\psi) = \boldsymbol{R}_x(\theta))$.

$$\boldsymbol{r}' = \boldsymbol{R}_{total}(\theta,\psi)\boldsymbol{r}'' = \boldsymbol{R}_x\,(\theta)\boldsymbol{r}''$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} +\sin\phi \\ -\sin\theta\cos\phi \\ +\cos\theta\cos\phi \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} +\sin\phi \\ -2\cos\theta\sin\theta\cos\phi \\ +(\cos^2\theta - \sin^2\theta)\cos\phi \end{bmatrix}$$

Post Axis Rotation Combined with Pitch Adjuster Tuning

Example: Reflected ray orientated via post axis rotation and mount adjuster tuning.

- Express the unit vector of the reflected ray in global coordinates using the $\boldsymbol{R}_y(\phi)$ matrix. Compare this result to those on Slide 25.

$$\boldsymbol{r} = \boldsymbol{R}_y(\phi)\boldsymbol{r}'$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} +\sin\phi \\ -2\cos\theta\sin\theta\cos\phi \\ +(\cos^2\theta - \sin^2\theta)\cos\phi \end{bmatrix} = \begin{bmatrix} +\cos\phi\sin\phi + (\cos^2\theta - \sin^2\theta)\cos\phi\sin\phi \\ -2\cos\theta\sin\theta\cos\phi \\ -\sin^2\phi + (\cos^2\theta - \sin^2\theta)\cos^2\phi \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos^2\theta + \sin^2\theta + \cos^2\theta - \sin^2\theta)\cos\phi\sin\phi \\ -2\cos\theta\sin\theta\cos\phi \\ -1 + (\cos^2\theta + \sin^2\theta)\cos^2\phi + (\cos^2\theta - \sin^2\theta)\cos^2\phi \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos^2\theta\cos\phi\sin\phi \\ -2\cos\theta\sin\theta\cos\phi \\ -1 + 2\cos^2\theta\cos^2\phi \end{bmatrix}$$

Post Axis Rotation Combined with Pitch Adjuster Tuning

Calculate arbitrary points (x_2, y_2, z_2) on the reflected ray.

- Start with the reflected ray's unit vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos^2\theta\cos\phi\sin\phi \\ -2\cos\theta\sin\theta\cos\phi \\ -1 + 2\cos^2\theta\cos^2\phi \end{bmatrix}$$

- Multiply the reflected ray's unit vector, $\langle x, y, z \rangle$, by a scaling factor (*A*). Add the result to the coordinates of a known point (x_1, y_1, z_1) on the reflected ray.

$$x_2 = x_1 + A(2\cos^2\theta\cos\phi\sin\phi)$$
$$y_2 = y_1 + A(-2\cos\theta\sin\theta\cos\phi)$$
$$z_2 = z_1 + A(-1 + 2\cos^2\theta\cos^2\phi)$$

- If the reflected ray's path began at the origin ($(x_1, y_1, z_1) = (0, 0, 0)$), points on the reflected ray can be calculated using the equations at the right and varying the scaling factor (A).

$$x_2 = A(2\cos^2\theta\cos\phi\sin\phi)$$
$$y_2 = A(-2\cos\theta\sin\theta\cos\phi)$$
$$z_2 = A(-1 + 2\cos^2\theta\cos^2\phi)$$